Computational analysis of information-processing properties in an echo-state network.

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Previous results

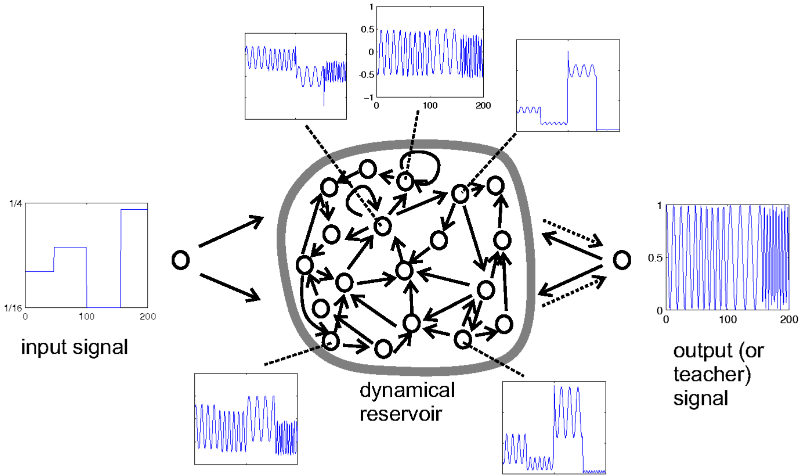
Artificial neural networks:

- Task performance is maximized when the network is operating in a state near the edge of chaos, or stability (Bertschinger and Natschläger, 2004)
- Information measures are maximized near the edge of stability (Boedecker et. al., 2012)

Biological neural networks:

- Cortical circuits may be tuned to criticality for optimized behavior (Beggs, 2008)

Echo state network:
Model



Echo state network
General model

Update equations:

$$\mathbf{x}(t) = \tanh(\mathbf{w}^{\mathrm{in}} u(t) + \mathbf{W}\mathbf{x}(t-1))$$

$$\mathbf{x}(t) \in \mathbb{R}^{N_x}, \mathbf{w}^{\mathrm{in}} \in \mathbb{R}^{N_x}, \mathbf{W} \in \mathbb{R}^{N_x \times N_x}$$

Output layer:

$$\mathbf{y}(t) = \mathbf{W}^{\mathrm{out}} \mathbf{x}(t)$$

$$\mathbf{y}(t) \in \mathbb{R}^{N_y}, \mathbf{W}^{\mathrm{out}} \in \mathbb{R}^{\mathbf{N_y} \times \mathbf{N_x}}$$

Learning:

$$\mathbf{W}^{\mathrm{out}} = \mathbf{U}\mathbf{X}^{+}$$

$$\mathbf{X}^{+} = \mathbf{X}^{\top}(\mathbf{X}\mathbf{X}^{\top})^{-1}$$

Information measures

Transfer entropy:

$$TE^{(k,l)}_{Y\to X} = I(X_t : \mathbf{Y}^{(l)}_{t-u} \mid \mathbf{X}^{(k)}_{t-1})$$

$$= \sum_{\mathbf{x}^{(k)}_{t-1},\mathbf{y}^{(l)}_{t-u}} p(\mathbf{x}^{(k)}_{t-1},\mathbf{y}^{(l)}_{t-u}) \sum_{x_t} p(x_t \mid \mathbf{x}^{(k)}_{t-1},\mathbf{y}^{(l)}_{t-u}) log_2 \frac{p(x_t \mid \mathbf{x}^{(k)}_{t-1},\mathbf{y}^{(l)}_{t-u})}{p(x_t \mid \mathbf{x}^{(k)}_{t-1})}$$

Active information storage:

$$AIS^{(k)}_X = I(X_t : \mathbf{X}^{(k)}_{t-1})$$

$$= \sum_{x_t,\mathbf{x}^{(k)}_{t-1}} p(x_t,\mathbf{x}^{(k)}_{t-1}) log_2 \frac{p(x_t,\mathbf{x}^{(k)}_{t-1})}{p(x_t)(\mathbf{x}^{(k)}_{t-1})}$$

where

$$\mathbf{X}^{(k)}_{t-1} = (X_{t-1}, X_{t-1-\tau_k}, X_{t-1-2\tau_k}, \ldots, , X_{t-1-(k-1)\tau_k})$$

$$\mathbf{Y}^{(l)}_{t-u} = (Y_{t-u}, Y_{t-u-\tau_l}, Y_{t-u-2\tau_l}, \ldots, , Y_{t-u-(l-1)\tau_l})$$

Transfer entropy:

$$\widehat{\mathrm{TE}}^{(k,l)}_{Y\to X} = \Psi(K)+\langle\Psi(n_{\mathbf{x}^{(k)}_{t-1}}+1)-\Psi(n_{x_t,\mathbf{x}^{(k)}_{t-1}}+1)-\Psi(n_{\mathbf{x}^{(k)}_{t-1},\mathbf{y}^{(l)}_{t-1}}+1)\rangle_t$$

where Ψ denotes the digamma function, $n_{(\cdot)}$ denotes the number of nearest neighbors in ϵ-hypercubes centered at $(\cdot)$ in the marginal spaces where ϵ is given by the Chebyshev distance of the realization $(x_t, \mathbf{x}^{(k)}_{t-1}, \mathbf{y}^{(l)}_{t-1})$ at time step t to its Kth nearest neighbor in the joint space, and $\langle\cdot\rangle_t$ denotes the time-average.

Active information storage:

$$\widehat{\mathrm{AIS}}^{(k)}_{X} = \Psi(K)+\Psi(N)-\langle\Psi(n_{\mathbf{x}^{(k)}_{t-1}}+1)+\Psi(n_{x_t}+1)\rangle_t$$

Let $\mathbf{x}(t+1) = \mathbf{f}(\mathbf{x}(t))$ be a discrete time dynamical system and $\mathbf{x}'(0) = \mathbf{x}(0) + \delta\mathbf{x}(0)$ be the initial conditions a trajectory $\mathbf{x}'(t)$ obtained by an infinitesimal displacement from $\mathbf{x}(0)$ such that $\gamma_0 = \|\delta\mathbf{x}(0)\| \ll 1$. Then the maximum Lyapunov characteristic exponent is

$$\lambda = \lim_{t\to\infty} \frac{1}{t} \ln\left(\frac{\gamma_t}{\gamma_0}\right) < \infty$$

where $\gamma_t = \|\mathbf{x}'(t) - \mathbf{x}(t)\|$.

$$\gamma_t \sim \gamma_0 e^{\lambda t} \Rightarrow \begin{cases} \lambda > 0 & \text{sensitive to initial conditions} \\ \lambda \approx 0 & \text{edge of criticality} \\ \lambda < 0 & \text{sub-critical} \end{cases}$$

Experimental setup
ESN model

- N=100 reservoir units
- single input unit
- output units
  - 120 for the MC task
  - single unit for the prediction task
- $w_j^{\mathrm{in}} \in \mathcal{U}(-0.1; 0.1), \forall j = 1, \ldots, N$
- $w_{ij}^{\mathrm{res}} \in \mathcal{N}(0; 0.5), \forall i, j = 1, \ldots, N$
- 100 samples transients
- 1000 samples training
- 2000 samples testing

Prediction tasks:

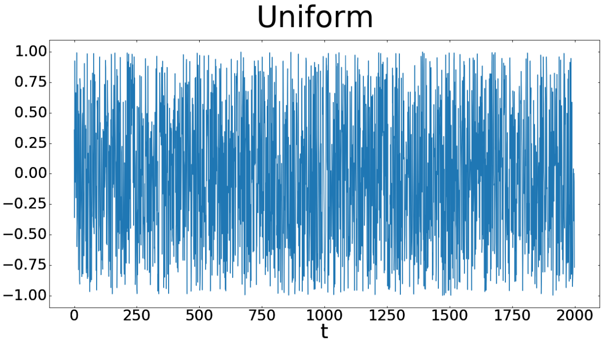
$$\mathrm{NRMSE} = \sqrt{\frac{\langle (\hat{y}(t) - y(t))^2 \rangle_t}{\langle (y(t) - \langle y(t) \rangle_t)^2 \rangle_t}}$$

MC task:

$$\mathrm{MC} = \sum_{q=1}^{q_{\max}} \mathrm{MC}_q = \sum_{q=1}^{q_{\max}} \frac{\mathrm{cov}^2(u(t-q), y_q(t))}{\mathrm{var}(u(t)) \cdot \mathrm{var}(y_q(t))}$$

Benchmark tasks
MC

$$\forall\, t : u(t) \sim \mathcal{U}(-1;1)$$

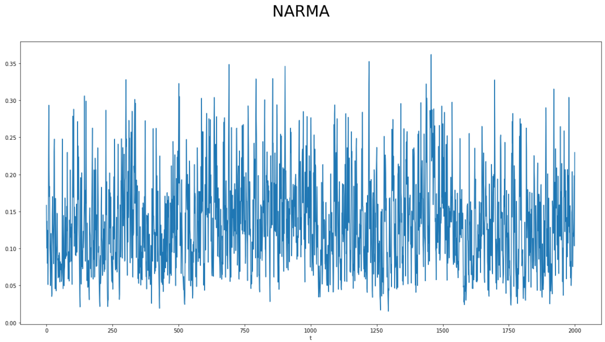


Benchmark tasks

NARMA

$$u(t+1) = 0.2\,u(t) + 0.004\,u(t)\sum_{i=0}^{29} u(t-i) + 1.5\,q(t-29)q(t) + 0.001$$

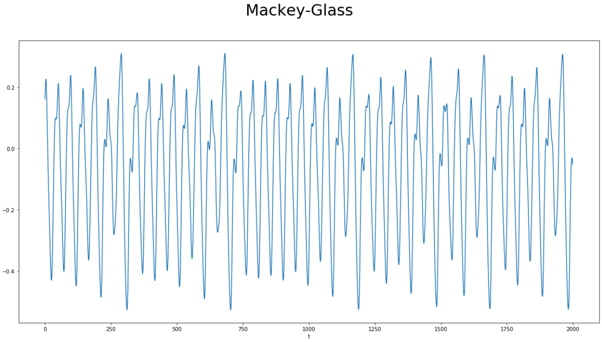
where $\forall\, t : q(t) \sim \mathcal{U}(0; 0.5)$



Benchmark tasks
Mackey-Glass system

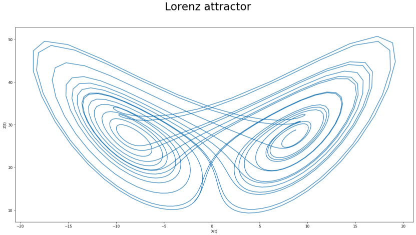
$$\frac{dy}{dt} = 0.2\frac{y_{17}}{1 + y_{17}^{10}} - 0.1y$$

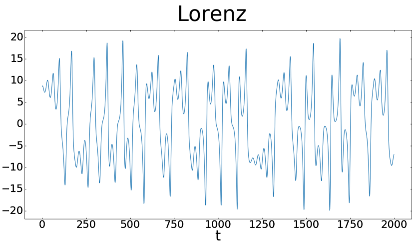
where u_{17} is the value of u at time $t - 17$.



Benchmark tasks
Lorenz system

$$\frac{dx}{dt} = 10(y - x)$$
$$\frac{dy}{dt} = x(28 - z) - y$$
$$\frac{dz}{dt} = xy - \frac{8}{3}z$$





ESN:

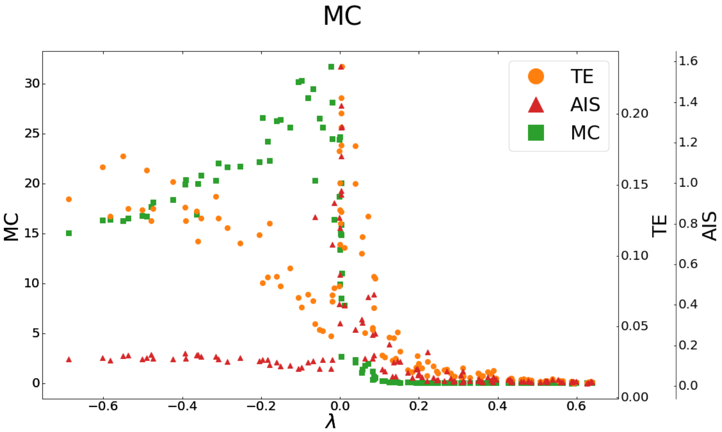
- fixed $\mathbf{w}^{\mathrm{in}}$
- $w_{ij}^{\mathrm{res}} \in \mathcal{N}(0;\sigma)$ such that $\log\sigma \in [-1.5;-0.25]$ in 26 steps 5 instances per value σ

Information measures:

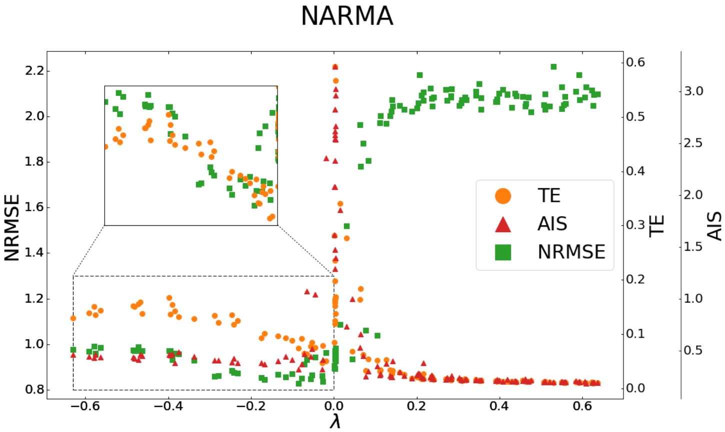
- TE
  - Target - $k = 2$, $\tau_k = 1$
  - Source - $l = 1$, $\tau_l = 1$
- AIS - $k = 2$, $\tau_k = 1$
- 4-NN

Information-theoretical measures around criticality

MC task

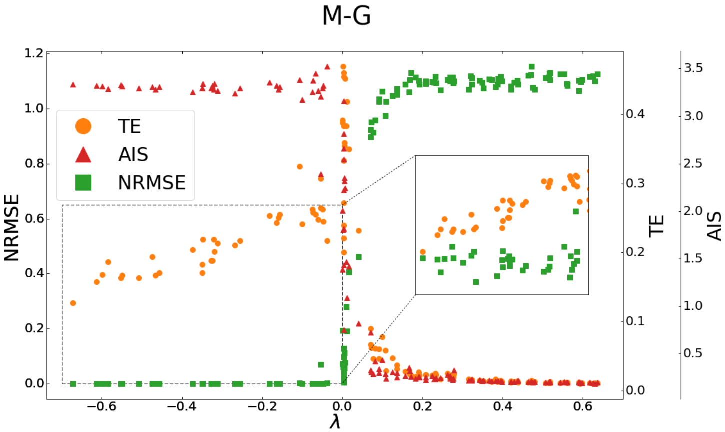


Information-theoretical measures around criticality
NARMA task



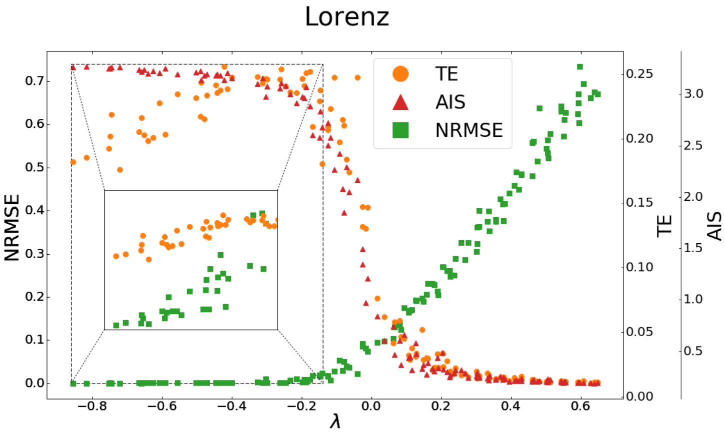
Information-theoretical measures around criticality

M–G task



Information-theoretical measures around criticality

Lorenz task



A closer look at information measures
Reservoir neuron signal embedding

Ragwitz-Kantz criterion:

$$(k, \tau) = \underset{k \in \mathcal{Z}, \tau \in \mathcal{Z}}{\text{argmin}} \|\mathbf{x} - \widehat{\mathbf{x}}(k, \tau)\|$$

where

$$\mathbf{x} = (x_1, \ldots, x_n), \ \hat{\mathbf{x}} = (\widehat{x}_1, \ldots, \widehat{x}_n)$$

Locally constant predictor:

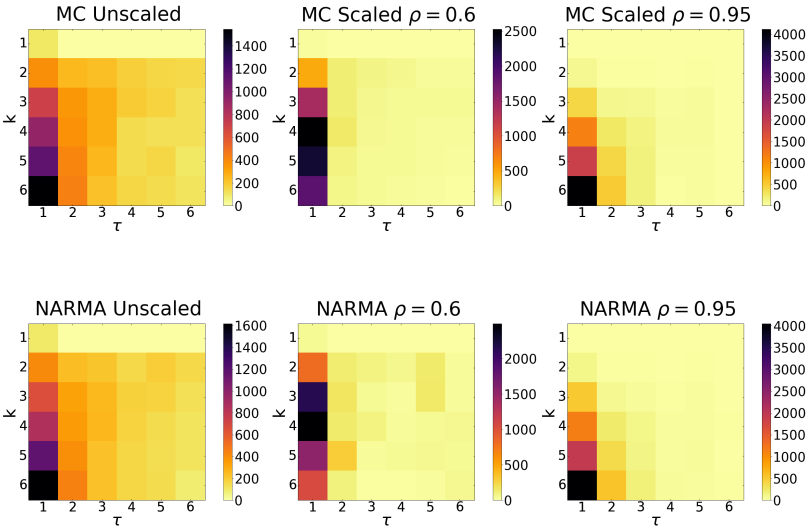
$$\widehat{x}_{t+1} = \frac{1}{\text{card}(\mathcal{U}_t)} \sum_{\mathbf{x_l} \in \mathcal{U}_t} x_{l+1}$$

$$\mathcal{U}_t = \{\mathbf{x}_l : \|\mathbf{x}_l - \mathbf{x}_t\| \leq \epsilon\}, \ \mathbf{x}_t = (x_t, x_{t-\tau_k}, x_{t-2\tau_k}, \ldots, , x_{t-(k-1)\tau})$$

- Ragwitz-Kantz search constrains:
  - $k = 1, \ldots, 6$
  - $\tau = 1, \ldots, 6$
- Reservoir spectral radius scalings:
  - Unstable: Unscaled
  - Stable: $\rho = 0.6$
  - Close to criticality: $\rho = 0.95$
- 100 instances per scaling and task

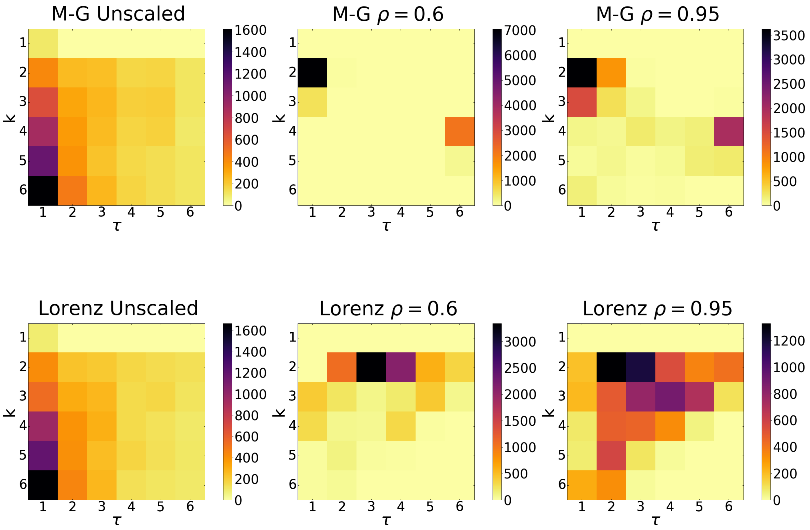
A closer look at information measures

Reservoir neuron signal embedding: MC and NARMA tasks



A closer look at information measures

Reservoir neuron signal embedding: M–G and Lorenz tasks



Most frequent (k, τ) pairs for each task and scaling

Scaling \ Task	MC		NARMA		M–G		Lorenz	
	k	τ	k	τ	k	τ	k	τ
init	6	1	6	1	6	1	6	1
$\rho = 0.6$	4	1	4	1	2	1	2	3
$\rho = 0.95$	6	1	6	1	2	1	2	2

A closer look at information measures

Experimental setup

ESN:

- 1 instance for all tasks and scalings
  - $w_j^{in} \in \mathcal{U}(-0.1; 0.1), \forall j = 1, \ldots, N$
  - $w_{ij}^{res} \in \mathcal{N}(0; 0.5), \forall i, j = 1, \ldots, N$
- Reservoir spectral radius scalings
  - Unstable: Unscaled
  - Stable: $\rho = 0.6$
  - Close to criticality: $\rho = 0.95$

Information measures:

- TE
  - Target - determined by Ragwitz-Kantz criterion
  - Source - $l = 1$, $\tau_l = 1$
- AIS - determined by Ragwitz-Kantz criterion
- 4-NN

$$H_0\colon T_{Y\to X} = 0 \text{ vs. } H_1\colon T_{Y\to X} > 0$$

p-value:

$$P(T_{Y_{sur}\to X} \geq T_{Y\to X}) = \sum_{Y_{sur}:\widehat{T}_{Y_{sur}\to X}\geq\widehat{T}_{Y\to X}} P(Y_{sur})$$

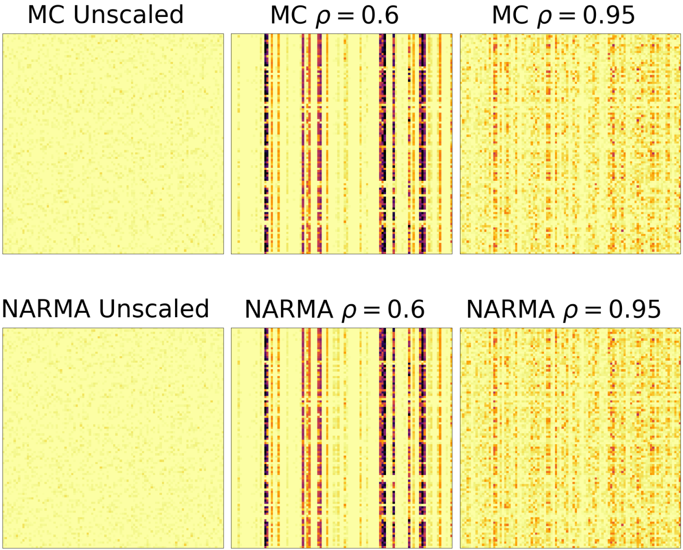
where Y_{sur} is permutaion of

$$Y = (\mathbf{y}_n^{(k)}, \mathbf{y}_{n-1}^{(k)}, \ldots, \mathbf{y}_1^{(k)}).$$

- p-value was computed from 100 $\widehat{T}_{Y_{sur}}$
- $\widehat{T}_{Y\to X} \leftarrow 0$ if $P(T_{Y_{sur}\to X} \geq T_{Y\to X}) > 0.05$

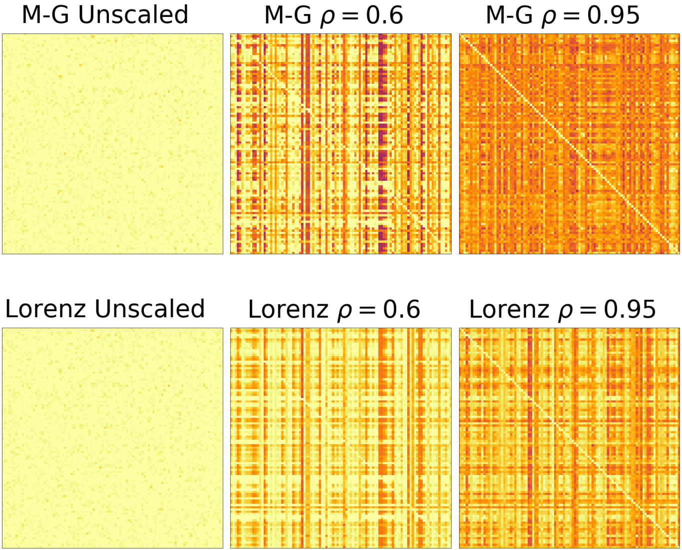
A closer look at information measures

TE matrices: MC and NARMA tasks



A closer look at information measures

TE matrices: M–G and Lorenz tasks

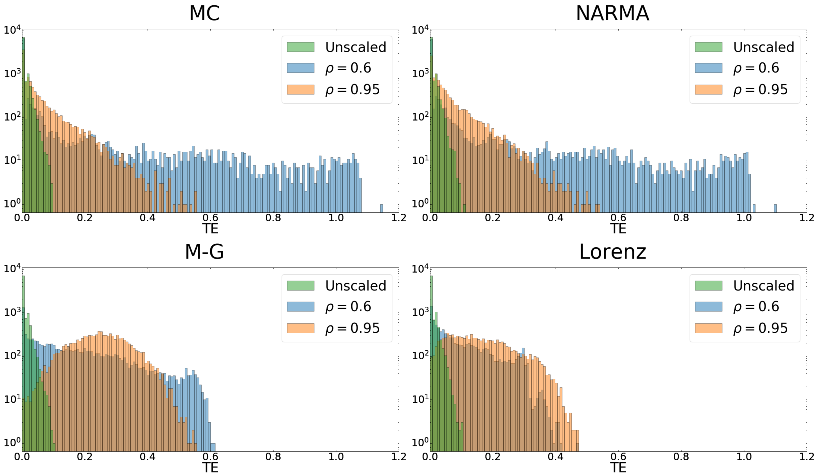


$$\left|\overline{H}(\mathrm{TE}_{\mathrm{RES}}^{(k,l)})\right| = \widehat{D_{\mathrm{KL}}}(\mathrm{TE}_{\mathrm{RES}}^{(k,l)} \,\|\, \mathcal{U}(0; \max \mathrm{TE}_{\mathrm{RES}}^{(k,l)}))$$
$$= \ln(\max \mathrm{TE}_{\mathrm{RES}}^{(k,l)}) - \widehat{H_{\mathrm{KL}}}(\mathrm{TE}_{\mathrm{RES}}^{(k,l)})$$

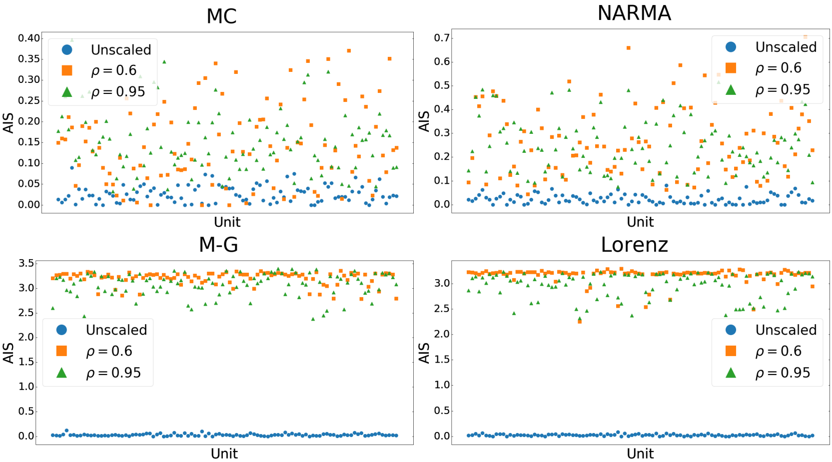
where $\ln(\max \mathrm{TE}_{\mathrm{RES}}^{(k,l)})$ is the exact differential entropy of the uniform distribution with the support in $[0; \max \mathrm{TE}_{\mathrm{RES}}^{(k,l)}]$, $\widehat{H_{\mathrm{KL}}}(\cdot)$ is the Kozachenko-Leonenko differential entropy estimator and $\mathrm{TE}_{\mathrm{RES}}^{(k,l)}$ is the distribution of the estimates of transfer entropies in the reservoir.

A closer look at information measures

TE distribution



A closer look at information measures
Active Information Storage



A closer look at information measures
Quantification of used measures

MC task	Unscaled	$\rho = 0.6$	$\rho = 0.95$
MC	0.06	17.8	32.8
Average TE	0.007	0.091	0.047
Average AIS	0.027	0.146	0.151
Rel. entr. of TE	0.981	0.629	1.147
LE	0.53	-0.52	-0.06

NARMA	Unscaled	$\rho = 0.6$	$\rho = 0.95$
NRMSE	2.143	0.98	0.83
Average TE	0.007	0.087	0.052
Average AIS	0.023	0.267	0.247
Rel. entr. of TE	1.086	0.783	1.174
LE	0.52	-0.53	-0.062

Mackey–Glass	Unscaled	$\rho = 0.6$	$\rho = 0.95$
NRMSE	1.13	0.00025	0.00026
Average TE	0.007	0.154	0.248
Average AIS	0.029	3.204	3.073
Rel. entr. of TE	0.266	0.386	0.587
LE	0.53	-0.52	-0.062

Lorenz	Unscaled	$\rho = 0.6$	$\rho = 0.95$
NRMSE	0.54	0.00012	0.0013
Average TE	0.007	0.087	0.157
Average AIS	0.025	3.157	2.971
Rel. entr. of TE	1.018	0.642	0.231
LE	0.52	-0.74	-0.28

Conclusion

- Two complexity classes in tested tasks
  - stochastic: MC/NARMA - benefit from criticality
  - deterministic: M–G/Lorenz - do not benefit from criticality
- Information measures are maximized at phase transition from stable to unstable regime
- Between complexity classes: increase in performance is associated with increase in global TE and AIS, and decrease of relative entropy of TE
- Within tasks: in stable regions there seems to be an inverse relationship between performance and global TE.
- Higher information flow does not mean better performance.
- Task specific

THANK YOU